Exercise 40

Use logarithmic differentiation to find the derivative of the function.

$$y = \frac{e^{-x}\cos^2 x}{x^2 + x + 1}$$

Solution

Take the natural logarithm of both sides and use the properties of logarithms to simplify the right side.

$$\begin{aligned}
\ln y &= \ln \frac{e^{-x}\cos^2 x}{x^2 + x + 1} \\
&= \ln\left(e^{-x}\cos^2 x\right) - \ln(x^2 + x + 1) \\
&= \ln e^{-x} + \ln\cos^2 x - \ln(x^2 + x + 1) \\
&= -x\ln e + \ln(\cos x)^2 - \ln(x^2 + x + 1) \\
&= -x + 2\ln|\cos x| - \ln(x^2 + x + 1)
\end{aligned}$$

Differentiate both sides with respect to x.

$$\begin{aligned}
\frac{d}{dx}(\ln y) &= \frac{d}{dx}[-x + 2\ln|\cos x| - \ln(x^2 + x + 1)] \\
\frac{1}{y}\cdot\frac{d}{dx}(y) &= -1 + \frac{2}{\cos x}\cdot\frac{d}{dx}(\cos x) - \frac{1}{x^2 + x + 1}\cdot\frac{d}{dx}(x^2 + x + 1) \\
\frac{1}{y}\cdot\frac{dy}{dx} &= -1 + \frac{2}{\cos x}\cdot(-\sin x) - \frac{1}{x^2 + x + 1}\cdot(2x + 1) \\
\frac{1}{y}\frac{dy}{dx} &= -1 - \frac{2\sin x}{\cos x} - \frac{2x + 1}{x^2 + x + 1} \\
\frac{dy}{dx} &= y\left[-\frac{2\sin x}{\cos x} + \frac{-2x - 1 - 1(x^2 + x + 1)}{x^2 + x + 1}\right] \\
&= \frac{e^{-x}\cos^2 x}{x^2 + x + 1}\left(-\frac{2\sin x}{\cos x} + \frac{-x^2 - 3x - 2}{x^2 + x + 1}\right) \\
&= -\frac{e^{-x}\cos^2 x}{x^2 + x + 1}\left[\frac{2\sin x}{\cos x} + \frac{x^2 + 3x + 2}{x^2 + x + 1}\right] \\
&= -\frac{e^{-x}\cos^2 x}{x^2 + x + 1}\left[2\tan x + \frac{(x + 2)(x + 1)}{x^2 + x + 1}\right]
\end{aligned}$$